Dimers and orthogonal polynomials: connections with random matrices

Extended lecture notes of the minicourse at IHP

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Abstract

In these lecture notes we present some connections between random matrices, the asymmetric exclusion process, random tilings. These three apparently unrelated objects have (sometimes) a similar mathematical structure, an interlacing structure, and the correlation functions are given in terms of a kernel. In the basic examples, the kernel is expressed in terms of orthogonal polynomials.

Contents

1 Gaussian Unitary Ensemble of random matrices (GUE)

Initially studied by statisticians in the 20's-30's, random matrices are then introduced in nuclear physics in the 60's to describe the energy levels distribution of heavy nuclei. The reader interested in a short discussion on random matrices in physics can read [37], in mathematics [43], and a good reference for a more extended analysis is [38].

1.1 The Gaussian Ensembles of random matrices

The Gaussian ensembles of random matrices have been introduced by physicists (Dyson, Wigner, ...) in the sixties to model statistical properties of the resonance spectrum of heavy nuclei. The energy levels of a quantum system are the eigenvalues of an Hamiltonian. For heavy nuclei some properties of the spectrum, like eigenvalues' spacing statistics, seemed to have some regularity (e.g., repulsions) common for all heavy nuclei. In other words there

was some universal behavior. If such a universal behavior exists, then it has to be the same as the behavior of choosing a random Hamiltonian. Moreover, since the heavy nuclei have a lot of bound stated (i.e., where the eigenvalues with normalizable eigenfunctions), the idea is to approximate it by a *large matrix* with random entries.

Even assuming the entries of the matrix random, to have a chance to describe the physical properties of heavy atoms, the matrices need to satisfy the intrinsic symmetries of the systems:

1. a *real symmetric* matrix can, a priori, describe a system with *time reversal symmetry* and *rotation invariance or integer magnetic momentum*,

2. a *real quaternionic* matrix (i.e., the basis are the Pauli matrices) can be used for *time reversal symmetry* and *half-integer magnetic momentum*,

3. a *complex hermitian* matrix can describe a system which is *not time reversal invariant* (e.g., with external magnetic field).

We now first focus on the complex hermitian matrices case, since it is the one we are going to discuss later on.

Definition 1. *The **Gaussian Unitary Ensemble** (GUE) of random matrices is a probability measure $\mathbb{P}$ on the set of $N \times N$ complex hermitian matrices, with density*

$$\mathbb{P}(\mathrm{d}H) = \frac{1}{Z_N} \exp\left(-\frac{\beta}{4N} \operatorname{Tr}(H^2)\right) \mathrm{d}H, \quad \textit{with } \beta = 2, \tag{1}$$

where $\mathrm{d}H = \prod_{i=1}^{N} \mathrm{d}H_{i,i} \prod_{1 \le i < j \le N} \mathrm{dRe}(H_{i,j}) \mathrm{dIm}(H_{i,j})$ is the reference measure, and Z_N is the normalization constant.

The meaning of $\beta = 2$ will be clear once we consider the induced measure on the eigenvalues.

Concerning the meaning of GUE, notice that the measure (1) is invariant over the unitary transformations and has a Gaussian form. The invariance over the unitary transformations is physically motivated, because the physical system does not depend on the choice of basis used to describe it. Indeed, by imposing that the measure $\mathbb{P}$ is (a) invariant under the change of basis (in the present case, invariant under the action of the group of symmetry $U(N)$) and

	$a = 1/2N$	$a = 1$	$a = N$
Largest eigenvalue	$2N + \mathcal{O}(N^{1/3})$	$\sqrt{2N} + \mathcal{O}(N^{-1/6})$	$\sqrt{2} + \mathcal{O}(N^{-2/3})$
Eigenvalues density	$\mathcal{O}(1)$	$\mathcal{O}(N^{1/2})$	$\mathcal{O}(N)$

Table 1: Typical scaling for the Gaussian Unitary Ensemble

(b) the entries of the matrices as independent random variables (of course, up to the required symmetry), then one gets the family of measures

$$\exp\left(-a\,\mathrm{Tr}(H^2) + b\,\mathrm{Tr}(H) + c\right), \quad a > 0, b, c \in \mathbb{R}. \tag{2}$$

The value of (c) is determined by the normalization requirement, while by an appropriate shift of the zeros of the energy (i.e., $H \to H - E$ for some given E), we can set $b = 0$. The energy shift is irrelevant from the physics point of view, because by the first principle of thermodynamics the energy of a system is an extensive observable defined *up to a constant*. The value of a remains free to be chosen. Different choices can be easily compared since are just a change of scale of each others. In the literature there are mainly three typical choices, see Table 1, each one with its own reason.

Another way to obtain (1) is to take the random variables, $H_{i,i} \sim \mathcal{N}(0, N)$ for $i = 1, \ldots, N$, and $\mathrm{Re}(H_{i,j}) \sim \mathcal{N}(0, N/2)$, $\mathrm{Im}(H_{i,j}) \sim \mathcal{N}(0, N/2)$ for $1 \leq i < j \leq N$ as independent random variables.

For the class of real symmetric (resp. quaternionic), one defines the Gaussian Orthogonal Ensemble (GOE) (resp. Gaussian Symplectic Ensemble (GSE)) as in Definition 1 but with $\beta = 1$ (resp. $\beta = 4$) and, of course, the reference measure is now the product Lebesgue measure over the independent entries of the matrices.

1.2 Eigenvalues' distribution

For the Gaussian ensembles of random matrices, one quantity of interest are the eigenvalues, which are independent of the choice of basis. By integrating out the angular variables one obtains the following result. Denote by $P_{\mathrm{GUE}}(\lambda)$ the probability density of eigenvalues at $\lambda \in \mathbb{R}^N$.

Proposition 2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}$ denote the N eigenvalues of a random matrix H with measure (1). Then, the joint distribution of eigenvalues is given by*

$$P_{\mathrm{GUE}}(\lambda) = \frac{1}{Z_N} |\Delta_N(\lambda)|^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4N}\lambda_i^2\right), \quad \text{with } \beta = 2, \tag{3}$$

$\Delta_N(\lambda) := \prod_{1 \le i < j \le N}(\lambda_j - \lambda_i)$ is the Vandermonde determinant, and Z_N is a normalization constant.

The following identity shows why Δ_N is called a determinant

$$\Delta_N(\lambda) = \det\left[\lambda_i^{j-1}\right]_{1 \le i,j \le N}. \tag{4}$$

Notice that $P_{\mathrm{GUE}}(\#\text{ e.v. } \in [x, x + \mathrm{d}x]) \sim (\mathrm{d}x)^2$, so that the probability of having double points is zero: it is a **simple point process**.

For GOE (resp. GSE) the joint distributions of eigenvalues has the form (3) but with $\beta = 1$ (resp. $\beta = 4$) instead.

1.3 Orthogonal polynomials

Here we describe orthogonal polynomials on $\mathbb{R}$, but the formulas can be easily adapted for polynomials on $\mathbb{Z}$ by replacing the Lebesgue measure by the counting measure and integrals by sums.

Definition 3. *Given a weight $\omega : \mathbb{R} \mapsto \mathbb{R}_+$, the **orthogonal polynomials** $\{q_k(x), k \ge 0\}$ are defined by the following two conditions,*

1. *$q_k(x)$ is a polynomial of degree k with $q_k(x) = u_k x^k + \ldots$, $u_k > 0$,*
2. *they satisfy the orthonormality condition,*

$$\langle q_k, q_l \rangle_\omega := \int_{\mathbb{R}} \omega(x) q_k(x) q_l(x) \mathrm{d}x = \delta_{k,l}. \tag{5}$$

Remark 4. There are other normalizations which are often used, like in the *Askey Scheme of hypergeometric orthogonal polynomials* [36]. Sometimes, the polynomials are taken to be *monic*, i.e., $u_k = 1$ and the orthonormality condition has then to be replaced by an orthogonality condition $\int_{\mathbb{R}} \omega(x)\tilde{q}_k(x)\tilde{q}_l(x)\mathrm{d}x = c_k\delta_{k,l}$. Of course $\tilde{q}_k(x) = q_k(x)/u_k$ and $c_k = 1/u_k^2$. Sometimes, the polynomials are neither orthonormal (like in Definition 3) nor monic, like the standard Hermite polynomials that we will encounter, and are given by derivatives of a given (generating) function.

A useful formula for sums of orthogonal polynomials is the **Christoffel-Darboux formula**:

$$\sum_{k=0}^{N-1} q_k(x)q_k(y) = \begin{cases} \dfrac{u_{N-1}}{u_N}\dfrac{q_N(x)q_{N-1}(y) - q_{N-1}(x)q_N(y)}{x-y}, & \text{for } x \neq y, \\ \dfrac{u_{N-1}}{u_N}(q'_N(x)q_{N-1}(x) - q'_{N-1}(x)q_N(x)), & \text{for } x = y. \end{cases} \tag{6}$$

This formula is proven by employing the following three term relation

$$q_n(x) = (A_n x + B_n)q_{n-1}(x) - C_n q_{n-2}(x), \tag{7}$$

with $A_n > 0, B_n, C_n > 0$ some constants. See Appendix B for details of the derivation. For polynomials as in Definition 3, it holds $A_n = u_n/u_{n-1}$ and $C_n = A_n/A_{n-1} = u_n u_{n-2}/u_{n-1}^2$.

1.4 Correlation functions of GUE

Now we restrict the point of view to the GUE ensemble and discuss the derivation of the correlation functions for the GUE eigenvalues' point process.

Let the reference measure be Lebesgue. Then, the n-point correlation function, $\rho^{(n)}_{\mathrm{GUE}}(x_1, \ldots, x_n)$ is the probability density of finding *an* eigenvalue *at each* of the x_k, $k = 1, \ldots, n$. P_{GUE} defined in (3) is symmetric in $\mathbb{R}^N$, which directly implies the following result.

Lemma 5. *The n-point correlation function for GUE eigenvalues is given by*

$$\rho^{(n)}_{\mathrm{GUE}}(x_1, \ldots, x_n) = \frac{N!}{(N-n)!}\int_{\mathbb{R}^{N-n}} P_{\mathrm{GUE}}(x_1, \ldots, x_N)\mathrm{d}x_{n+1}\ldots\mathrm{d}x_N \tag{8}$$

for $n = 1, \ldots, N$ and $\rho^{(n)}_{\mathrm{GUE}}(x_1, \ldots, x_n) = 0$ for $n > N$.

It is important to notice that it is not fixed which eigenvalue is at which position. In particular, we have

$$\rho^{(1)}_{\mathrm{GUE}}(x) = \text{eigenvalues' density at } x \quad \Longrightarrow \quad \int_{\mathbb{R}} \rho^{(1)}_{\mathrm{GUE}}(x)\mathrm{d}x = N. \tag{9}$$

More generally,

$$\int_{\mathbb{R}^n} \rho^{(n)}_{\mathrm{GUE}}(x_1, \ldots, x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n = \frac{N!}{(N-n)!}. \tag{10}$$

Our next goal is to do the integration in (8). For that purpose, we will use Hermite orthogonal polynomials. For any family of polynomials $\{q_k, k = 0, \dots, N-1\}$, where q_k has degree k, by multi-linearity of the determinant, it holds

$$\Delta_N(\lambda) = \det[\lambda_i^{j-1}]_{1\le i,j\le N} = \text{const} \times \det[q_{j-1}(\lambda_i)]_{1\le i,j\le N}. \tag{11}$$

Therefore, setting $\omega(x) := \exp(-x^2/2N)$, we have

$$\begin{aligned} &P_{\text{GUE}}(\lambda_1, \dots, \lambda_N) \\ &= \text{const} \times \det[q_{k-1}(\lambda_i)]_{1\le i,k\le N} \det[q_{k-1}(\lambda_j)]_{1\le k,j\le N} \prod_{i=1}^N \omega(\lambda_i) \\ &= \text{const} \times \det\left[\sum_{k=1}^N q_{k-1}(\lambda_i) q_{k-1}(\lambda_j)\right]_{1\le i,j\le N} \prod_{i=1}^N \omega(\lambda_i). \end{aligned} \tag{12}$$

Notice that until this point, the polynomials q's do not have to be orthogonal. However, if we choose the polynomials orthogonal with respect to the weight ω, then the integrations in (8) becomes particularly simple and nice.

Proposition 6. *Let q_k be orthogonal polynomials with respect to the weight $\omega(x) = \exp(-x^2/2N)$. Then,*

$$\rho^{(n)}_{\text{GUE}}(x_1, \dots, x_n) = \det\left[K_N^{\text{GUE}}(x_i, x_j)\right]_{1\le i,j\le n}, \tag{13}$$

where

$$K_N^{\text{GUE}}(x, y) = \sqrt{\omega(x)}\sqrt{\omega(y)} \sum_{k=0}^{N-1} q_k(x) q_k(y). \tag{14}$$

The proof of Proposition 6 can be found in Appendix C. What we need to do is integrate out the $N - n$ variables one after the other and see that the determinant keeps the same entries but becomes smaller. For that we use the following two identities

$$\begin{aligned} \int_{\mathbb{R}} K_N^{\text{GUE}}(x, x) \mathrm{d}x &= N, \\ \int_{\mathbb{R}} K_N^{\text{GUE}}(x, z) K_N^{\text{GUE}}(z, y) \mathrm{d}z &= K_N^{\text{GUE}}(x, y). \end{aligned} \tag{15}$$

It is at this point that the particular choice of orthogonal polynomials is essential. Indeed, (15) hold for the kernel defined by the r.h.s. of (14) precisely when $q_k(x)$ are taken to be orthogonal to $\omega(x)$.

The particular form of correlation functions (13) appears also in other systems.

Definition 7. *A point process (i.e., a random point measure) is called* ***determinantal*** *if its n-point correlation function has the form*

$$\rho^{(n)}(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{1 \le i,j \le n} \tag{16}$$

for some (measurable) function $K : \mathbb{R}^2 \to \mathbb{R}$, called the ***kernel*** *of the determinantal point process.*

One might ask when a measure defines a determinantal point process. A sufficient condition is the following (see Proposition 2.2 of [4]).

Theorem 8. *Consider a probability measure on $\mathbb{R}^N$ of the form*

$$\frac{1}{Z_N} \det[\phi_i(x_j)]_{1 \le i,j \le N} \det[\psi_i(x_j)]_{1 \le i,j \le N} \prod_{i=1}^{N} \omega(x_i) \mathrm{d}x_i, \tag{17}$$

with the normalization $Z_N \neq 0$. Then (17) defines a determinantal point process with kernel

$$K_N(x, y) = \sum_{i,j=1}^{N} \psi_i(x) [A^{-1}]_{i,j} \phi_j(y), \tag{18}$$

where $A = [A_{i,j}]_{1 \le i,j \le N}$,

$$A_{i,j} = \langle \phi_i, \psi_j \rangle_\omega = \int_{\mathbb{R}} \omega(z) \phi_i(z) \psi_j(z) \mathrm{d}z. \tag{19}$$

1.5 GUE kernel and Hermite polynomials

Finally, let us give an explicit formula for the kernel K_N. The **standard Hermite polynomials**, $\{H_n, n \ge 0\}$, are defined by

$$H_k(y) = (-1)^k e^{y^2} \frac{\mathrm{d}^k}{\mathrm{d}y^k} e^{-y^2}. \tag{20}$$

They satisfy

$$\int_{\mathbb{R}} e^{-y^2} H_k(y) H_l(y) \mathrm{d}y = \sqrt{\pi} 2^k k! \delta_{k,l}, \tag{21}$$

with $H_k(y) = 2^k y^k + \ldots$, and also

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-y^2} H_n(y)\right) = -e^{-y^2} H_{n+1}(y) \quad \Rightarrow \quad \int_{-\infty}^{x} e^{-y^2} H_{n+1}(y) \mathrm{d}y = -e^{-x^2} H_n(x). \tag{22}$$

By the change of variable $y = x/\sqrt{2N}$ and a simple computation, one shows that

$$q_k(x) = \frac{1}{\sqrt[4]{2\pi N}} \frac{1}{\sqrt{2^k k!}} H_k\left(\frac{x}{\sqrt{2N}}\right) \tag{23}$$

are orthogonal polynomials with respect to $\omega(x) = \exp(-x^2/2N)$, and that $u_k = (2\pi N)^{-1/4} k!^{-1/2} N^{-k/2}$. Then, Christoffel-Darboux formula (6) gives

$$K_N^{\mathrm{GUE}}(x,y) = \begin{cases} \dfrac{q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y)}{x-y} N e^{-(x^2+y^2)/4N}, & \text{for } x \neq y, \\ (q_N'(x) q_{N-1}(x) - q_{N-1}'(x) q_N(x)) N e^{-(x^2+y^2)/4N}, & \text{for } x = y. \end{cases} \tag{24}$$

With our normalization in (1) the eigenvalues' density remains bounded and the largest eigenvalue is around the value $2N$. Indeed, the eigenvalues' density at position μN is given by

$$\rho^{(1)}(\mu N) = K_N^{\mathrm{GUE}}(\mu N, \mu N) \overset{N\to\infty}{\longrightarrow} \begin{cases} \dfrac{1}{\pi}\sqrt{1-(\mu/2)^2}, & \text{for } \mu \in [-2,2], \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

The large-N asymptotic density in (25) is called **Wigner's semicircle law**.

1.6 Distribution of the largest eigenvalue: gap probability

How to compute the distribution of the largest eigenvalue, $\lambda_{\max}$? One uses the following simple relation,

$$\mathbb{P}(\lambda_{\max} \leq s) = \mathbb{P}(\text{no eigenvalue lies in } (s, \infty)). \tag{26}$$

This is a special case of **gap probability**, i.e., probability that in a Borel set B there are no eigenvalues. The gap probabilities are expressed in terms of n-point correlation functions as follows.

$$\begin{aligned}
&\mathbb{P}(\text{no eigenvalue lies in } B) = \mathbb{E}\Big(\prod_i (1 - \mathbb{1}_B(\lambda_i))\Big) \\
&= \sum_{n\geq 0}(-1)^n \mathbb{E}\Big(\sum_{i_1<\ldots<i_n}\prod_{k=1}^n \mathbb{1}_B(\lambda_{i_k})\Big) \overset{\text{sym}}{=} \sum_{n\geq 0}\frac{(-1)^n}{n!}\mathbb{E}\Big(\sum_{\substack{i_1,\ldots,i_n \\ \text{all different}}}\prod_{k=1}^n \mathbb{1}_B(\lambda_{i_k})\Big) \\
&= \sum_{n\geq 0}\frac{(-1)^n}{n!}\int_{B^n}\rho^{(n)}(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n,
\end{aligned} \tag{27}$$

where $\mathbb{1}_B(x) = 1$ if $x \in B$ and $\mathbb{1}_B(x) = 0$ if $x \notin B$. The last step holds for simple point processes.

For a determinantal point process, like the GUE eigenvalues, the r.h.s. of (27) is the Fredholm series expansion of the Fredholm determinant:

$$\begin{aligned}
\mathbb{P}(\lambda_{\max} \leq s) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{(s,\infty)^n}\det[K_N^{\mathrm{GUE}}(x_i,x_j)]_{1\leq i,j\leq n}\mathrm{d}x_1\ldots\mathrm{d}x_n \\
&\equiv \det(\mathbb{1} - K_N^{\mathrm{GUE}})_{L^2((s,\infty),\mathrm{d}x)}.
\end{aligned} \tag{28}$$

In our case the sums over n are actually finite (for $n > N$ the terms are equal to zero, since the kernel K_N has rank N), but we preferred to keep the formulation of the general case.

1.7 Correlation functions of GUE minors: interlacing structure

Interestingly, the determinantal structure still holds in the enlarged setting of eigenvalues of minors.

Consider a $N \times N$ GUE random matrix H and denote by $\lambda_1^N, \ldots, \lambda_N^N$ its eigenvalues. Then, denote by H_m the $m \times m$ minor of the matrix H where the last $N - m$ rows and columns have been deleted. Denote by $\lambda_1^m, \ldots, \lambda_m^m$ the eigenvalues of H_m. In [26, 33] the correlation functions of $\{\lambda_k^m, 1 \leq k \leq m \leq N\}$ are determined. It turns out that also in that case the correlation functions are determinantal (on $\{1, \ldots, N\} \times \mathbb{R}$).

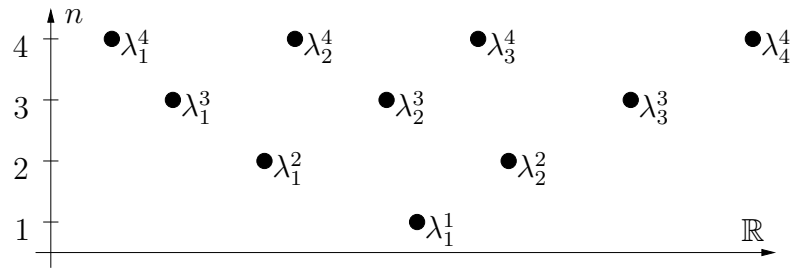

Figure 1: Illustration of the interlacing structure of the GUE minors' eigenvalues.

Let us order the eigenvalues for each minors as follows, $\lambda_1^m \leq \lambda_2^m \leq \ldots \leq \lambda_m^m$. Then, the GUE minor measure can be written as, see e.g. [26],

$$\text{const} \times \left(\prod_{m=1}^{N-1} \det[\phi(\lambda_i^m, \lambda_j^{m+1})]_{1 \leq i,j \leq m+1} \right) \det[\Psi_{N-i}^N(\lambda_j^N)]_{1 \leq i,j \leq N} \tag{29}$$

where $\lambda_{m+1}^m \equiv \text{virt}$ are *virtual variables*, $\phi(x,y) = \mathbb{1}(x > y)$, $\phi(\text{virt}, y) = 1$, and

$$\Psi_{N-k}^N(x) = \frac{(-1)^{N-k}}{(2N)^{(N-k)/2}} e^{-x^2/2N} H_{N-k}(x/\sqrt{2N}). \tag{30}$$

The eigenvalues of the minors have a nice structure which appears in other problems too, see next sections. (29) is non-zero if and only if the eigenvalues satisfy the following **interlacing structure**, see Figure 1 for an illustration,

$$\lambda_1^{m+1} < \lambda_1^m \leq \lambda_2^{m+1} < \lambda_2^m \leq \ldots < \lambda_m^m \leq \lambda_{m+1}^{m+1}, \tag{31}$$

for $m = 1, \ldots, N-1$. Indeed,

$$\det[\phi(\lambda_i^m, \lambda_j^{m+1})]_{1 \leq i,j \leq m+1} = \begin{cases} 1, & \text{if (31) is satisfied,} \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

One might have seen that some inequalities were replaced by strict inequalities, but this is irrelevant since the events with $\lambda_k^n = \lambda_k^{n+1}$ have probability zero.

Let us define,

$$\Psi_{n-k}^n(x) := (\phi * \Psi_{n+1-k}^{n+1})(x). \tag{33}$$

Then, a straightforward consequence of (22), is

$$\Psi^n_{n-k}(x) = \frac{(-1)^{n-k}}{(2N)^{(n-k)/2}} e^{-x^2/2N} H_{n-k}(x/\sqrt{2N}) \tag{34}$$

for $1 \le k \le n$. Moreover, it is easy to verify that

$$\int_{\mathbb{R}^{n(n-1)/2}} \prod_{1\le k\le m\le n-1} \mathrm{d}\lambda^m_k \prod_{m=1}^{n-1} \det[\phi(\lambda^m_i, \lambda^{m+1}_j)]_{1\le i,j\le m+1} = \Delta_n(\lambda^n_1, \dots, \lambda^n_n). \tag{35}$$

To get an explicit expression of the kernel (see Theorem 10 below), we need to find functions $\{\Phi^n_{n-k}, k = 1, \dots, n\}$ orthogonal, with respect to the weight $\omega = 1$, to the functions $\{\Psi^n_{n-j}, j = 1, \dots, n\}$, and such that

$$\mathrm{span}\{\Phi^n_0(x), \dots, \Phi^n_{n-1}(x)\} = \mathrm{span}\{1, x, \dots, x^{n-1}\}. \tag{36}$$

We find

$$\Phi^n_{n-j}(x) = \frac{(-1)^{n-j}}{\sqrt{2\pi}(n-j)!} \left(\frac{N}{2}\right)^{(n-j)/2} H_{n-j}(x/\sqrt{2N}). \tag{37}$$

A measure of the form (29) has determinantal correlations [9]. The difference with the above case is that the correlation functions are determinantal on $\{1, \dots, N\} \times \mathbb{R}$ instead of $\mathbb{R}$. This means the following. The probability density of finding an eigenvalue of H_{n_i} at position x_i, for $i = 1, \dots, n$, is given by

$$\rho^{(n)}((n_i, x_i), 1 \le i \le n) = \det\left[K^{\mathrm{GUE}}_N(n_i, x_i; n_j, x_j)\right]_{1\le i,j\le n}. \tag{38}$$

The kernel K^{GUE}_N is computed applying the general Theorem 10 which we report below, but first we give the solution.

Proposition 9. *The correlation functions of the GUE minors are determinantal with kernel given by*

$$K^{\mathrm{GUE}}_N(n_1, x_1; n_2, x_2) = -(\phi^{*(n_2-n_1)})(x_1, x_2)\mathbb{1}_{[n_1<n_2]} + \sum_{k=1}^{n_2} \Psi^{n_1}_{n_1-k}(x_1)\Phi^{n_2}_{n_2-k}(x_2), \tag{39}$$

with $\phi^{*(n_2-n_1)}$ the convolution of ϕ with itself $n_2 - n_1$ times, namely, for $n_2 > n_1$,

$$(\phi^{*(n_2-n_1)})(x_1, x_2) = \frac{(x_2 - x_1)^{n_2-n_1-1}}{(n_2 - n_1 - 1)!} \mathbb{1}_{[x_2-x_1\ge 0]}. \tag{40}$$

The following result is generalization of Theorem 8.

Theorem 10. *Assume we have a measure on* $\{x_i^n, 1 \le i \le n \le N\}$ *given in the form,*

$$\frac{1}{Z_N}\Big(\prod_{n=1}^{N-1} \det[\phi_n(x_i^n, x_j^{n+1})]_{1\le i,j\le n+1}\Big) \det[\Psi_{N-i}^N(x_j^N)]_{1\le i,j\le N}, \tag{41}$$

where x_{n+1}^n *are some "virtual" variables and* Z_N *is a normalization constant. If* $Z_N \neq 0$*, then the correlation functions are determinantal.*

To write down the kernel we need to introduce some notations. Define

$$\phi^{(n_1,n_2)}(x,y) = \begin{cases} (\phi_{n_1} * \cdots * \phi_{n_2-1})(x,y), & n_1 < n_2, \\ 0, & n_1 \ge n_2, \end{cases} \tag{42}$$

where $(a*b)(x,y) = \int_{\mathbb{R}} a(x,z)b(z,y)\mathrm{d}z$*, and, for* $1 \le n < N$*,*

$$\Psi_{n-j}^n(x) := (\phi^{(n,N)} * \Psi_{N-j}^N)(y), \quad j = 1, 2, \ldots, N. \tag{43}$$

Set $\phi_0(x_1^0, x) = 1$*. Then the functions*

$$\{(\phi_0 * \phi^{(1,n)})(x_1^0, x), \ldots, (\phi_{n-2} * \phi^{(n-1,n)})(x_{n-1}^{n-2}, x), \phi_{n-1}(x_n^{n-1}, x)\} \tag{44}$$

are linearly independent and generate the n*-dimensional space* V_n*. Define a set of functions* $\{\Phi_j^n(x), j = 0, \ldots, n-1\}$ *spanning* V_n *defined by the orthogonality relations*

$$\int_{\mathbb{R}} \Phi_i^n(x)\Psi_j^n(x)\mathrm{d}x = \delta_{i,j} \tag{45}$$

for $0 \le i, j \le n-1$*.*

Under Assumption (A): $\phi_n(x_{n+1}^n, x) = c_n \Phi_0^{(n+1)}(x)$*, for some* $c_n \neq 0$*,* $n = 1, \ldots, N-1$*, the kernel takes the simple form*

$$K(n_1, x_1; n_2, x_2) = -\phi^{(n_1,n_2)}(x_1, x_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(x_1)\Phi_{n_2-k}^{n_2}(x_2). \tag{46}$$

Remark 11. Without Assumption (A), the correlations functions are still determinantal but the formula is modified as follows. Let M be the $N \times N$ dimensional matrix defined by $[M]_{i,j} = (\phi_{i-1} * \phi^{(i,N)} * \Psi_{N-j}^N)(x_i^{i-1})$. Then

$$\begin{aligned} & K(n_1, x_1; n_2, x_2) \qquad (47) \\ = \; & -\phi^{(n_1,n_2)}(x_1, x_2) + \sum_{k=1}^{N} \Psi_{n_1-k}^{n_1}(x_1) \sum_{l=1}^{n_2} [M^{-1}]_{k,l} (\phi_{l-1} * \phi^{(l,n_2)})(x_l^{l-1}, x_2). \end{aligned}$$

The "virtual variables" are just there to write the formula in a simpler way, but they do not represent real variables. This can be seen for example in Assumption (A), where the r.h.s. does not depend on x^n_{n+1}. Theorem 10 is proven by using the framework of [17].

In the case of the measure (29), the n-dimensional space V_n is spanned by $\{1, x, \dots, x^{n-1}\}$, so Φ^n_k have to be polynomials of degree k, compare with (37).

2 Totally Asymmetric Simple Exclusion Process (TASEP)

2.1 Continuous time TASEP: interlacing structure

The totally asymmetric simple exclusion process (TASEP) is one of the simplest interacting stochastic particle systems. It consists of particles on the lattice of integers, $\mathbb{Z}$, with *at most one particle at each site* (exclusion principle). The dynamics in continuous time is as follows. Particles jump on the neighboring right site with rate 1 provided that the site is empty. This means that jumps are independent of each other and take place after an exponential waiting time with mean 1, which is counted from the time instant when the right neighbor site is empty.

Here we consider all particles with equal rate 1. However, the framework which we explain below, can be generalized to particle-dependent rates and also particles can jump on both directions (but with different rules, it is not the partially asymmetric case): by jumping on their right, particles can be blocked, while on the left if a site is occupied, then it is pushed by the jumping particle. This generalization, called PushASEP, together with a partial extension to space-time correlations is the content of our paper [6].

On the macroscopic level the particle density, $u(x,t)$, evolves deterministically according to the Burgers equation $\partial_t u + \partial_x(u(1-u)) = 0$ [49]. Therefore a natural question is to focus on fluctuation properties, which exhibit rather unexpected features. The asymptotic results can be found in the literature, see Appendix A. Here we focus on a method which can be used to analyze the joint distributions of particles' positions. This method is based on a interlacing structure first discovered by Sasamoto in [51], later extended and generalized in a series of papers, starting with [9]. We explain the key steps

following the notations of [9], where the details of the proofs can be found.

Consider the TASEP with N particles starting at time $t = 0$ at positions $y_N < \ldots < y_2 < y_1$. The first step is to obtain the probability that at time t these particles are at positions $x_N < \ldots < x_2 < x_1$, which we denote by

$$G(x_1, \ldots, x_N; t) = \mathbb{P}((x_N, \ldots, x_1; t)|(y_N, \ldots, y_1; 0)). \tag{48}$$

This function has firstly been determined using Bethe-Ansatz method [53]. A posteriori, the result can be checked directly by writing the master equation.

Lemma 12. *The transition probability is given by*

$$G(x_1, \ldots, x_N; t) = \det(F_{i-j}(x_{N+1-i} - y_{N+1-j}, t))_{1 \leq i,j \leq N} \tag{49}$$

with

$$F_n(x, t) = \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{\mathrm{d}w}{w} \frac{(1-w)^{-n}}{w^{x-n}} e^{t(w-1)}, \tag{50}$$

where $\Gamma_{0,1}$ is any simple loop oriented anticlockwise which includes $w = 0$ and $w = 1$.

The key property of Sasamoto's decomposition is the following relation

$$F_{n+1}(x, t) = \sum_{y \geq x} F_n(y, t). \tag{51}$$

Denote $x_1^k := x_k$ the position of TASEP particles. Using the multi-linearity of the determinant and (51) one obtains

$$G(x_1, \ldots, x_N; t) = \sum_{\mathcal{D}'} \det(F_{-j+1}(x_i^N - y_{N-j+1}, t))_{1 \leq i,j \leq N}, \tag{52}$$

where

$$\mathcal{D}' = \{x_i^n, 2 \leq i \leq n \leq N | x_i^n \geq x_{i-1}^{n-1}\}. \tag{53}$$

Then, using the antisymmetry of the determinant and Lemma 13 below we can rewrite (52) as

$$G(x_1, \ldots, x_N; t) = \sum_{\mathcal{D}} \det(F_{-j+1}(x_i^N - y_{N-j+1}, t))_{1 \leq i,j \leq N}, \tag{54}$$

where

$$\mathcal{D} = \{x_i^n, 2 \leq i \leq n \leq N | x_i^n > x_i^{n+1}, x_i^n \geq x_{i-1}^{n-1}\}. \tag{55}$$

Lemma 13. *Let f an antisymmetric function of $\{x_1^N, \dots, x_N^N\}$. Then, whenever f has enough decay to make the sums finite,*

$$\sum_{\mathcal{D}} f(x_1^N, \dots, x_N^N) = \sum_{\mathcal{D}'} f(x_1^N, \dots, x_N^N), \tag{56}$$

with the positions $x_1^1 > x_1^2 > \ldots > x_1^N$ being fixed.

Now, notice that, for $n = -k < 0$, (50) has only a pole at $w = 0$, which implies that

$$F_{n+1}(x,t) = -\sum_{y<x} F_n(x,t). \tag{57}$$

Define then

$$\begin{aligned} \Psi_k^N(x) &:= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w} \frac{(1-w)^k}{w^{x-y_{N-k}-k}} e^{t(w-1)}, \\ \phi(x,y) &:= \mathbb{1}(x>y), \quad \phi(\mathrm{virt}, y) = 1. \end{aligned} \tag{58}$$

In particular, for $k = 1, \dots, N$, $\Psi_k^N(x) = (-1)^k F_{-k}(x - y_{N-k}, t)$. Then,

$$\det[\phi(x_i^n, x_j^{n+1})]_{1\le i,j\le n+1} = \begin{cases} 1, & \text{if (60) is satisfied,} \\ 0, & \text{otherwise,} \end{cases} \tag{59}$$

where the **interlacing condition** $\mathcal{D}$, is given by

$$x_1^{n+1} < x_1^n \le x_2^{n+1} < x_2^n \le \ldots < x_n^n \le x_{n+1}^{n+1}, \tag{60}$$

for $n = 1, \dots, N-1$, see Figure 2 for a graphical representation.

Therefore, we can replace the sum over $\mathcal{D}$ by a product of determinants of increasing sizes. Namely,

$$G(x_1, \dots, x_N; t) = \sum_{\substack{x_k^n \in \mathbb{Z} \\ 2\le k\le n\le N}} Q(\{x_k^n, 1 \le k \le n \le N\}) \tag{61}$$

where the measure Q is

$$\begin{aligned} Q(\{x_k^n, 1 \le k \le n \le N\}) = \Big(\prod_{n=1}^{N-1} \det(\phi(x_i^n, x_j^{n+1}))_{1\le i,j\le n+1} \Big) \\ \times \det(\Psi_{N-j}^N(x_i^N))_{1\le i,j\le N}, \end{aligned} \tag{62}$$

where we set $x_{n+1}^n = \mathrm{virt}$ (we call them virtual variables). As for the GUE minor case, if we compute $\Psi_{n-k}^n(x) := (\phi * \Psi_{n+1-k}^{n+1})(x)$, we get the (58) but with N replaced by n.

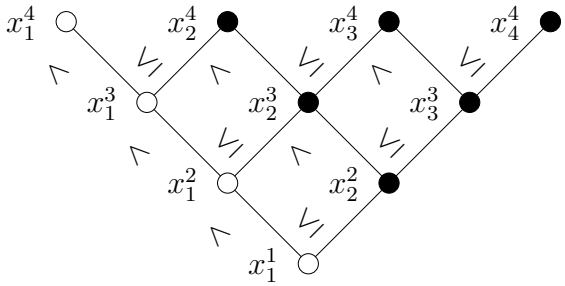

Figure 2: Graphical representation of the domain of integration $\mathcal{D}$ for $N = 4$. One has to "integrate" out the variables x_i^j, $i \geq 2$ (i.e., the black dots). The positions of x_1^k, $k = 1, \ldots, N$ are given (i.e., the white dots).

Remark 14. The measure (62) is not necessarily a probability measure, since positivity is not ensured, but still has the same structure as the GUE minors measure, compare with (29). In particular, the distribution of TASEP particles' position can be expressed in the same form as the gap probability, but in general for a signed determinantal point process. The precise statement is in Theorem 15 below.

Applying the discrete version of Theorem 10, see Lemma 3.4 of [9], we get the following result.

Theorem 15. *Let us start at time $t = 0$ with N particles at positions $y_N < \ldots < y_2 < y_1$. Let $\sigma(1) < \sigma(2) < \ldots < \sigma(m)$ be the indices of m out of the N particles. The joint distribution of their positions $x_{\sigma(k)}(t)$ is given by*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m} \big\{x_{\sigma(k)}(t) \geq s_k\big\}\Big) = \det(\mathbb{1} - \chi_s K_t \chi_s)_{\ell^2(\{\sigma(1),\ldots,\sigma(m)\}\times\mathbb{Z})} \tag{63}$$

where $\chi_s(\sigma(k), x) = \mathbb{1}(x < s_k)$. K_t is the extended kernel with entries

$$K_t(n_1, x_1; n_2, x_2) = -\phi^{(n_1,n_2)}(x_1, x_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(x_1)\Phi_{n_2-k}^{n_2}(x_2) \tag{64}$$

where

$$\phi^{(n_1,n_2)}(x_1, x_2) = \binom{x_1 - x_2 - 1}{n_2 - n_1 - 1}, \tag{65}$$

$$\Psi_i^n(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w^{i+1}} \frac{(1-w)^i}{w^{x-y_{n-i}}} e^{t(w-1)}, \tag{66}$$

and the functions $\Phi_i^n(x)$, $i = 0, \ldots, n-1$, *form a family of polynomials of degree* $\leq n$ *satisfying*

$$\sum_{x \in \mathbb{Z}} \Psi_i^n(x) \Phi_j^n(x) = \delta_{i,j}. \tag{67}$$

The path Γ_0 *in the definition of* Ψ_i^n *is any simple loop, anticlockwise oriented, which includes the pole at* $w = 0$ *but* not *the one at* $w = 1$.

We omit to write explicitly the dependence on the set $\{y_i\}$ is hidden in the definition of the Φ_i^n's and the Ψ_i^n's.

2.2 Correlation functions for step initial conditions: Charlier polynomials

Now we consider the particular case of step initial conditions, $y_k = -k$ for $k \geq 1$. In that case, the measure (62) is positive, i.e., it is a probability measure. The correlation functions of subsets of $\{x_k^n(t), 1 \leq k \leq n, n \geq 1\}$ are determinantal with kernel K_t^{TASEP}, which is computed by Theorem 15.

The correlation kernel K_t^{TASEP} is given in terms of Charlier polynomials, which we now introduce. The Charlier polynomial of degree n is denoted by $C_n(x,t)$ and defined as follows. Consider the weight ω on $\mathbb{Z}_+ = \{0, 1, \ldots\}$

$$\omega(x) = e^{-t} t^x / x!. \tag{68}$$

Then, $C_n(x,t)$ is defined via the orthogonal condition

$$\sum_{x \geq 0} \omega(x) C_n(x,t) C_m(x,t) = \frac{n!}{t^n} \delta_{n,m} \tag{69}$$

or, equivalently, $C_n(x,t) = (-1/t)^n x^n + \cdots$. They can be expressed in terms of hypergeometric functions

$$C_n(x,t) = {}_2F_0(-n, -x; \, ; -1/t) = C_x(n,t). \tag{70}$$

From the generating function of the Charlier polynomials

$$\sum_{n \geq 0} \frac{C_n(x,t)}{n!} (tw)^n = e^{wt} (1-w)^x \tag{71}$$

one gets the integral representation

$$\frac{t^n}{n!}C_n(x,t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{e^{wt}(1-w)^x}{w^{n+1}}. \tag{72}$$

Equations (70) and (72) then give

$$\Psi_k^n(-n+x) = \frac{e^{-t}t^x}{x!}C_k(x,t) \quad \text{and} \quad \Phi_j^n(-n+x) = \frac{t^j}{j!}C_j(x,t). \tag{73}$$

In particular, the kernel of the joint distributions of $\{x_1^N, \dots, x_N^N\}$ is

$$\begin{aligned} K_t^{\mathrm{TASEP}}(N, -N+x; N, -N+y) &= \sum_{k=0}^{N-1} \Psi_k^N(-N+x)\Phi_k^N(-N+y) \\ &= \omega(x) \sum_{k=0}^{N-1} q_k(x)q_k(y) \end{aligned} \tag{74}$$

with $q_k(x) = (-1)^k \frac{t^{k/2}}{\sqrt{k!}} C_k(x,t) = (t^k k!)^{-1/2} x^k + \dots$, orthogonal polynomials with respect to $\omega(x)$. Therefore, using the Christoffel-Darboux formula (6) we get, for $x \neq y$,

$$\begin{aligned} &K_t^{\mathrm{TASEP}}(N, -N+x; N, -N+y) \\ &\qquad = \frac{e^{-t}t^x}{x!} \frac{t^N}{(N-1)!} \frac{C_{N-1}(x,t)C_N(y,t) - C_N(x,t)C_{N-1}(y,t)}{x-y}. \end{aligned} \tag{75}$$

Remark 16. The expression of the kernel in (74) is not symmetric in x and y. Is this wrong? No! For a determinantal point process the kernel is not uniquely defined, but only up to conjugation. For any function $f > 0$, the kernel $\widetilde{K}(x,y) = f(x)K(x,y)f(y)^{-1}$ describes the same determinantal point process of the kernel $K(x,y)$. Indeed, the f cancels out in the determinants defining the correlation functions. In particular, by choosing $f(x) = 1/\sqrt{\omega(x)}$, we get a symmetric version of (74).

Double integral representation of K_t^{TASEP}

Another typical way of representing the kernel K_t^{TASEP} is as double integral. This representation is well adapted to large-t asymptotic analysis (both in the bulk and the edge).

Let us start with (66) with $y_k = -k$. We have

$$\Psi_k^n(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{e^{t(w-1)}(1-w)^k}{w^{x+n+1}}, \tag{76}$$

and remark that $\Psi_k^n(x) = 0$ for $x < -n$, $k \geq 0$. The orthogonal functions $\Phi_k^n(x)$, $k = 0, \ldots, n-1$, have to be polynomials of degree k and to satisfy the orthogonal relation $\sum_{x \geq -n} \Psi_k^n(x)\Phi_j^n(x) = \delta_{i,j}$. They are given by

$$\Phi_j^n(x) = \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z \frac{e^{-t(z-1)} z^{x+n}}{(1-z)^{j+1}}. \tag{77}$$

Indeed, for any choice of the paths Γ_0, Γ_1 such that $|z| < |w|$,

$$\begin{aligned}
\sum_{x \geq -n} \Psi_k^n(x)\Phi_j^n(x) &= \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,z}} \mathrm{d}w \frac{e^{t(w-1)}}{e^{t(z-1)}} \frac{(1-w)^k}{(1-z)^{j+1}} \sum_{x \geq -n} \frac{z^{x+n}}{w^{x+n+1}} \\
&= \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,z}} \mathrm{d}w \frac{e^{t(w-1)}}{e^{t(z-1)}} \frac{(1-w)^k}{(1-z)^{j+1}} \frac{1}{w-z} \\
&= \frac{(-1)^{k-j}}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z (z-1)^{k-j-1} = \delta_{j,k},
\end{aligned} \tag{78}$$

since the only pole in the last w-integral is the simple pole at $w = z$.

After the orthogonalization, we can determine the kernel. Let us compute the last term in (64). For $k > n_2$, we have $\Phi_{n_2-k}^{n_2}(x) = 0$. So, by choosing z close enough to 1, we have $|1-z| < |1-w|$, and we can extend the sum over k to $+\infty$, i.e.,

$$\begin{aligned}
\sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(x_1)\Phi_{n_2-k}^{n_2}(x_2) &= \sum_{k=1}^{n_2} \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{tw} z^{x_2+n_2}}{e^{tz} w^{x_1+n_1+1}} \frac{(1-w)^{n_1-k}}{(1-z)^{n_2-k+1}} \\
&= \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{tw} z^{x_2+n_2}}{e^{tz} w^{x_1+n_1+1}} \sum_{k=1}^{n_2} \frac{(1-w)^{n_1-k}}{(1-z)^{n_2-k+1}} \\
&= \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{tw} z^{x_2+n_2}}{e^{tz} w^{x_1+n_1+1}} \sum_{k=1}^{\infty} \frac{(1-w)^{n_1-k}}{(1-z)^{n_2-k+1}} \\
&= \frac{-1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{tw} (1-w)^{n_1} z^{x_2+n_2}}{e^{tz} (1-z)^{n_2} w^{x_1+n_1+1}} \frac{1}{z-w}.
\end{aligned} \tag{79}$$

The integral over Γ_1 contains only the pole $z = 1$, while the integral over Γ_0 only the pole $w = 0$ (i.e., $w = z$ is not inside the integration paths). This is the kernel for $n_1 \geq n_2$. For $n_1 < n_2$, there is the extra term $-\phi^{(n_1,n_2)}(x_1, x_2)$. It is not difficult to check that

$$\begin{aligned} &\frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_w} \mathrm{d}z \frac{e^{tw}(1-w)^{n_1} z^{x_2+n_2}}{e^{tz}(1-z)^{n_2} w^{x_1+n_1+1}} \frac{1}{z-w} \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{(1-w)^{n_1-n_2}}{w^{x_1+n_1-(x_2+n_2)+1}} = \binom{x_1 - x_2 - 1}{n_2 - n_1 - 1}. \end{aligned} \tag{80}$$

Therefore, the double integral representation of K_t^{TASEP} (for step initial condition) is the following:

$$\begin{aligned} &K_t^{\mathrm{TASEP}}(n_1, x_1; n_2, x_2) \\ &= \begin{cases} \dfrac{-1}{(2\pi \mathrm{i})^2} \displaystyle\oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{tw}(1-w)^{n_1} z^{x_2+n_2}}{e^{tz}(1-z)^{n_2} w^{x_1+n_1+1}} \frac{1}{z-w}, & \text{if } n_1 \geq n_2, \\ \dfrac{-1}{(2\pi \mathrm{i})^2} \displaystyle\oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_{1,w}} \mathrm{d}z \frac{e^{tw}(1-w)^{n_1} z^{x_2+n_2}}{e^{tz}(1-z)^{n_2} w^{x_1+n_1+1}} \frac{1}{z-w}, & \text{if } n_1 < n_2. \end{cases} \end{aligned} \tag{81}$$

Remark 17. Notice that in the GUE case, the most natural object are the eigenvalues for the $N \times N$ matrix, $\lambda_1^N, \ldots, \lambda_N^N$. They are directly associated with a determinantal point process. The corresponding quantities in terms of TASEP are not the positions of the particles, $x_1^1, \ldots, x_1^N$. The measure on these particle positions is not determinantal, but with the extension to the larger picture, namely to $\{x_k^n, 1 \leq k \leq n \leq N\}$, we recover a determinantal structure. This is used to determine the joint distributions of our particles, since in terms of $\{x_k^n, 1 \leq k \leq n \leq N\}$ we need only to compute a gap probability.

2.3 Discrete time TASEP

There are several discrete time dynamics of TASEP from which the continuous time limit can be obtained. The most common dynamics are:

- **Parallel update**: at time $t \in \mathbb{Z}$ one first selects which particles can jump (their right neighboring site is empty). Then, the configuration of particles at time $t + 1$ is obtained by moving independently with probability $p \in (0, 1)$ the selected particles.

- **Sequential update**: one updates the particles from right to left. The configuration at time $t+1$ is obtained by moving with probability $p \in (0,1)$ the particles whose right rite is empty. This procedure is from right to left, which implies that also a block of m particles can move in one time-step with probability p^m.

Other dynamical rules have also been introduced, see [54] for a review.

Sequential update

For the TASEP with sequential update, there is an analogue of Lemma 12, with the only difference lying in the functions F_n (see [48] and Lemma 3.1 of [8]). The functions F_n satisfy again the recursion relation (51) and Theorem 15 still holds with the only difference being in the $\Psi_i^n(x)$'s which are now given by

$$\Psi_i^n(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w^{i+1}} \frac{(1-p+pw)^t (1-w)^i}{w^{x-y_{n-i}}}. \tag{82}$$

For step initial conditions, the kernel is then given by (81) with e^{tw}/e^{tz} replaced by $(1-p+pw)^t/(1-p+pz)^t$.

Parallel update

For the TASEP with parallel update, the same formalism used above can still be applied. However, the interlacing condition and the transition functions ϕ_n are different. The details can be found in Section 3 of [10] (in that paper we also consider the case of different times, which can be used for example to study the tagged particle problem). The analogue of (62) is the following (see Proposition 7 of [10]).

Lemma 18. *The transition probability $G(x;t)$ can be written as a sum over*

$$\mathcal{D}'' = \{x_i^n, 2 \le i \le n \le N | x_i^n > x_{i-1}^n\} \tag{83}$$

as follows:

$$G(x_1, \dots, x_N; t) = \sum_{\mathcal{D}''} \widetilde{Q}(\{x_k^n, 1 \le k \le n \le N\}), \tag{84}$$

where

$$\widetilde{Q}(\{x_k^n, 1 \le k \le n \le N\}) = \Big(\prod_{n=1}^{N-1} \det(\phi^\sharp(x_{i-1}^n, x_j^{n+1}))_{1\le i,j\le n+1} \Big) \times \det(F_{-j+1}(x_i^N - y_{N-j+1}, t+1-j))_{1\le i,j\le N}. \tag{85}$$

where we set $x_0^n = -\infty$ *(we call them virtual variables). The function* $\phi^\sharp$ *is defined by*

$$\phi^\sharp(x,y) = \begin{cases} 1, & y \ge x, \\ p, & y = x-1 \\ 0, & y \le x-2, \end{cases} \tag{86}$$

and F_{-n} *is given by*

$$F_{-n}(x,t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,-1}} \mathrm{d}w \frac{w^n}{(1+w)^{x+n+1}} (1+pw)^t. \tag{87}$$

The product of determinants in (85) also implies a **weighted interlacing condition**, different from the one of TASEP. More precisely,

$$x_i^{n+1} \le x_i^n - 1 \le x_{i+1}^{n+1}. \tag{88}$$

The difference between the continuous-time TASEP is that it can happens that $x_{i+1}^{n+1} = x_i^n - 1$. However, for each occurrence of such a configuration, the weight is multiplied by p. The continuous-time limit is obtained by replacing t by t/p and letting $p \to 0$.

Also in this case Theorem 15 holds with the following new functions,

$$\begin{aligned} \widetilde{\phi}^{(n_1,n_2)}(x_1,x_2) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,-1}} \mathrm{d}w \frac{1}{(1+w)^{x_1-x_2+1}} \left(\frac{w}{(1+w)(1+pw)} \right)^{n_1-n_2} \mathbb{1}_{[n_2>n_1]}, \\ \Psi_i^n(x) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,-1}} \mathrm{d}w \frac{(1+pw)^t}{(1+w)^{x-y_{n-i}+1}} \left(\frac{w}{(1+w)(1+pw)} \right)^i, \end{aligned} \tag{89}$$

and Φ_k^n are polynomials of degree k given by the orthogonality condition (67): $\sum_{x\in\mathbb{Z}} \Psi_i^n(x)\Phi_j^n(x) = \delta_{i,j}$. In particular, for step initial conditions, $y_k = -k$, $k \ge 1$, we obtain the following result.

Proposition 19. *The correlation kernel for discrete-time TASEP with parallel update is given by*

$$K_t^{\mathrm{TASEP}}(n_1,x_1;n_2,x_2) = -\phi^{(n_1,n_2)}(x_1,x_2) + \widetilde{K}_t^{\mathrm{TASEP}}(n_1,x_1;n_2,x_2), \tag{90}$$

with $\phi^{(n_1,n_2)}$ *given by*

$$\phi^{(n_1,n_2)}(x_1,x_2) = \frac{\mathbb{1}_{[n_2>n_1]}}{2\pi \mathrm{i}} \oint_{\Gamma_{-1}} \mathrm{d}w \frac{(1+pw)^{n_2-n_1} w^{n_1-n_2}}{(1+w)^{(x_1+n_1)-(x_2+n_2)+1}} \tag{91}$$

and

$$\widetilde{K}_t^{\mathrm{TASEP}}(n_1,x_1;n_2,x_2) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{-1}} \mathrm{d}z \frac{(1+pw)^{t-n_1+1}}{(1+pz)^{t-n_2+1}} \frac{w^{n_1}(1+z)^{x_2+n_2}}{z^{n_2}(1+w)^{x_1+n_1+1}} \frac{1}{w-z}. \tag{92}$$

Remark 20. The discrete-time parallel update TASEP with step initial condition is equivalent to the shuffling algorithm on the Aztec dynamics as shown in [41]. This particle dynamics also fits in the framework developed in [5]. See [21] for an animation, where the particles have coordinates $(z_i^n := x_i^n + n, n)$.

3 $2+1$ dynamics: connection to random tilings and random matrices

In recent years there has been a lot of progress in understanding large time fluctuations of driven interacting particle systems on the one-dimensional lattice. Evolution of such systems is commonly interpreted as random growth of a one-dimensional interface, and if one views the time as an extra variable, the evolution produces a random surface (see e.g. Figure 4.5 in [44] for a nice illustration). In a different direction, substantial progress have also been achieved in studying the asymptotics of random surfaces arising from dimers on planar bipartite graphs.

Although random surfaces of these two kinds were shown to share certain asymptotic properties (also common to random matrix models), no direct connection between them was known. We present a class of two-dimensional random growth models (that is, the principal object is a randomly growing surface, embedded in the four-dimensional space-time).

In two different projections these models yield random surfaces of the two kinds mentioned above (one reduces the spatial dimension by one, the second projection is to fixced time).

Let us now we explain the $2+1$-dimensional dynamics. Consider the set of variables $\{x_k^n(t), 1 \le k \le n, n \ge 1\}$ and let us see what is their evolution inherited from the TASEP dynamics.

3.1 $2+1$ dynamics for continuous time TASEP

Consider continuous time TASEP with step-initial condition, $y_k = -k$ for $k \geq 1$. Then, a further property of the measure (62) with step initial conditions is that

$$x_k^n(0) = -n + k - 1. \tag{93}$$

Let us verify it. The first $N-1$ determinants in (62) imply the interlacing condition (60). In particular, $x_1^N(0) = -N$ and $x_k^N(0) \geq -N + k - 1$ for $k \geq 2$. At time $t = 0$ we have

$$F_{-k}(x, 0) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{(w-1)^k}{w^{x+k+1}}. \tag{94}$$

By residue's theorem, we have $F_{-k}(x, 0) = 0$ for $x \geq 1$ (no pole at ∞), $F_{-k}(x, 0) = 0$ for $x < -k$ (no pole at 0), and $F_{-k}(0, 0) = 1$.

The last determinant in (62) is then given by the determinant of

$$\begin{bmatrix} F_0(0,0) & F_{-1}(-1,0) & \cdots & F_{-N+1}(-N+1,0) \\ F_0(x_2^N(0)+N,0) & F_{-1}(x_2^N(0)+N-1) & \cdots & F_{-N+1}(x_2^N(0)+1,0) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(x_N^N(0)+N,0) & F_{-1}(x_N^N(0)+N-1,0) & \cdots & F_{-N+1}(x_N^N(0)+1,0), \end{bmatrix}. \tag{95}$$

Let us see when $\det(95) \neq 0$:

1. since $x_k^N(0) \geq -N + k - 1$, the first column of (95) is $[1, 0, \ldots, 0]^t$.
2. Then, if $x_2^N(0) > -N + 1$, the second column is $[*, 0, \ldots, 0]^t$ and $\det(95) = 0$. Thus we have $x_2^N(0) = -N + 1$.
3. Repeating the argument for the other columns, we obtain that $\det(95) \neq 0$ if and only if $x_k^N(0) = -N + k - 1$ for $k = 3, \ldots, N$.

This initial condition is illustrated in Figure 3 (top, left).

Now we explain the dynamics on the variables $\{x_k^n(t), 1 \leq k \leq n, n \geq 1\}$ which is inherited by the dynamics on the TASEP particles $\{x_1^n(t), n \geq 1\}$. Each of the particles x_k^m has an independent exponential clock of rate one, and when the x_k^m-clock rings the particle attempts to jump to the right by one. If at that moment $x_k^m = x_k^{m-1} - 1$ then the jump is blocked. If that is not the case, we find the largest $c \geq 1$ such that $x_k^m = x_{k+1}^{m+1} = \cdots = x_{k+c-1}^{m+c-1}$, and all c particles in this string jump to the right by one.

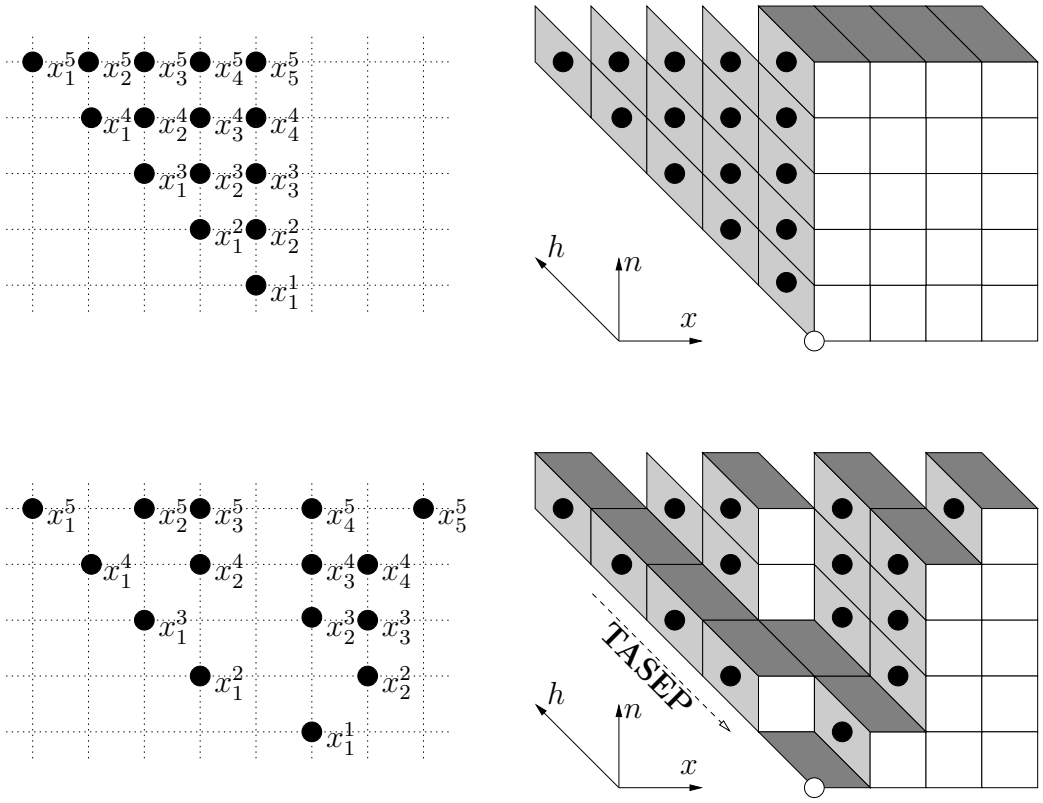

Figure 3: (top, left) Illustration of the initial conditions for the particles system. (bottom, left) A configuration obtained from the initial conditions. (right) The corresponding lozenge tiling configurations. In the height function picture, the white circle has coordinates $(x, n, h) = (-1/2, 0, 0)$. For a Java animation of the model see [20].

Informally speaking, the particles with smaller upper indices are heavier than those with larger upper indices, so that the heavier particles block and push the lighter ones in order for the interlacing conditions to be preserved.

We illustrate the dynamics using Figure 3, which shows a possible configuration of particles obtained from our initial condition. If in this state of the system the x_1^3-clock rings, then particle x_1^3 does not move, because it is blocked by particle x_1^2. If it is the x_2^2-clock that rings, then particle x_2^2 moves to the right by one unit, but to keep the interlacing property satisfied, also particles x_3^3 and x_4^4 move by one unit at the same time. This aspect of the dynamics is called "pushing".

3.2 Interface growth interpretation

Figure 3 (right) has a clear three-dimensional connotation. Given the random configuration $\{x_k^n(t)\}$ at time moment t, define the random *height function*

$$\begin{aligned} &h : (\mathbb{Z}+\tfrac{1}{2}) \times \mathbb{Z}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}, \\ &h(x,n,t) = \#\{k \in \{1,\dots,n\} \mid x_k^n(t) > x\}. \end{aligned} \tag{96}$$

In terms of the tiling on Figure 3, the height function is defined at the vertices of rhombi, and it counts the number of particles to the right from a given vertex. (This definition differs by a simple linear function of (x,n) from the standard definition of the height function for lozenge tilings, see e.g. [34,35].) The initial condition corresponds to starting with perfectly flat facets.

In terms of the step surface of Figure 3, the evolution consists of removing all columns of (x,n,h)-dimensions $(1,*,1)$ that could be removed, independently with exponential waiting times of mean one. For example, if x_2^2 jumps to its right, then three consecutive cubes (associated to x_2^2, x_3^3, x_4^4) are removed. Clearly, in this dynamics the directions x and n do not play symmetric roles. Indeed, this model belongs to the $2+1$ anisotropic KPZ class of stochastic growth models, see [5, 7].

3.3 Random tilings interpretation

A further interpretation of our particles' system is a random tiling model. To see that one surrounds each particle location by a rhombus of one type (the light-gray in Figure 3) and draws edges through locations where there are no particles. In this way we have a random tiling with three types of tiles that

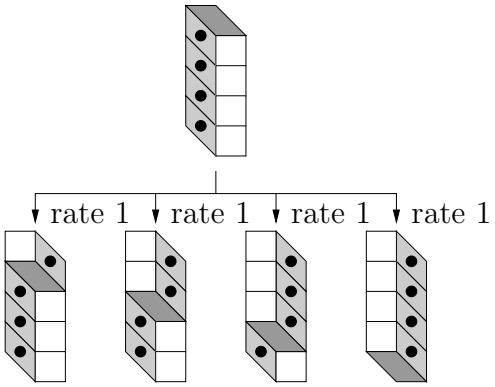

Figure 4: Illustration of the dynamics on tiles for a column of height $m = 4$.

we call *white*, *light-gray*, and *dark-gray*. Our initial condition corresponds to a perfectly regular tiling.

The dynamics on random tilings is the following. Consider all sub-configuration of the random tiling which looks like a visible column, i.e., for some $m \geq 1$, there are m light-gray tiles on the left of m white tiles (and then automatically closed by a dark-gray tile). The dynamics is an exchange of light-gray and white tiles within the column. More precisely, for a column of height m, for all $k = 1, \ldots, m$, independently and with rate 1, there is an exchange between the top k light-gray tiles with the top white tiles as illustrated in Figure 4 for the case $m = 4$.

Remark 21. We can also derive a determinantal formula not only for the correlation of light-gray tiles, but also for the three types of tiles. This is explicitly stated in Theorem 5.2 of [5].

3.4 Diffusion scaling and relation with GUE minors

There is an interesting *partial* link with GUE minors. In the diffusion scaling limit

$$\xi_k^n := \sqrt{2N} \lim_{t \to \infty} \frac{x_k^n(t) - t}{\sqrt{2t}} \tag{97}$$

the measure on $\{\xi_k^n, 1 \leq k \leq n \leq N\}$ is exactly given by (29).

Remark 22. It is important to stress, that this correspondence is a fixed-time result. From this, a dynamical equivalence does not follow. Indeed, if

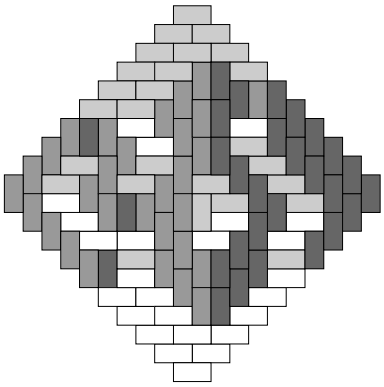

Figure 5: A random tiling of the Aztec diamond of size $n = 10$.

we let the GUE matrices evolves according to the so-called Dyson's Brownian Motion dynamics, then the evolution of the minors is not the same as the (properly rescaled) evolution from our $2 + 1$ dynamics for TASEP.

3.5 Shuffling algorithm and discrete time TASEP

An Aztec diamond is a shape like the outer border of Figure 5. The shuffling algorithm provides a way of generating after n steps uniform sample of an Aztec diamond of size n tiled with dominos.

We now discuss (not prove) the connection between discrete time TASEP with parallel update and step initial condition. Moreover, we take the parameter $p = 1/2$ to get uniform distribution. It is helpful to do a linear change of variable. Instead of x_k^n we use

$$z_k^n = x_k^n + n, \tag{98}$$

so that the interlacing condition becomes

$$z_k^{n+1} \leq z_k^n \leq z_{k+1}^{n+1}. \tag{99}$$

The step initial condition for TASEP particles is $z_1^n(0) = 0$, $n \geq 1$. An analysis similar to the one of Section 3.1 leads to $z_k^n(0) = k - 1$, $1 \leq k \leq n$. Then, the dynamics on $\{z_k^n, 1 \leq k \leq n, n \geq 1\}$ inherited by discrete time parallel update TASEP is the following. First of all, during the time-step from $n - 1$ to n, all particles with upper-index greater or equal to $n + 1$ are

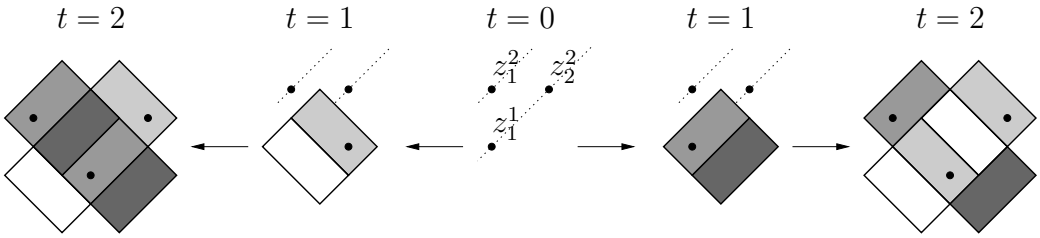

Figure 6: Two examples of configurations at time $t = 2$ obtained by the particle dynamics and its associated Aztec diamonds (rotated by 45 degrees).

frozen. Then, from level n down to level 1, particles independently jump on their right site with probability 1/2, provided the interlacing condition (99) with the lower levels are satisfied. If the interlacing condition would be violated for particles in upper levels, then these particles are also pushed by one position to restore (99).

Finally, let us explain how to associate a tiling configuration to a particle configuration. Up to time $t = n$ only particles with upper-index at most n could have possibly moved. These are also the only particles which are taken into account to determine the random tiling. The tiling follows these rules, see Figure 6 for an illustration:

1. light-gray tiles: placed on each particle which moved in the last time-step,

2. middle-gray tiles: placed on each particle which did not move in the last time-step,

3. dark-gray tiles and white tiles: in the remaining position, depending on the tile orientation.

The proof of the equivalence of the dynamics can be found in [41], where particle positions are slightly shifted with respect to Figure 6. In [21] you can find a Java animation of the dynamics.

A Further references

In this section we give further references, in particular, of papers based on the approach described in this lecture notes.

- *Interlacing structure and random matrices*: In [33] it is studied the GUE minor process which arises also in the Aztec diamond at the turning points. Turning points and GUE minor process occurs also for some class of Young diagrams [42]. The antisymmetric version of the GUE minors is studied in [25]. In [26] the correlation functions for several random matrix ensembles are obtained, using two methods: the interlacing structure from [9] and the approach of [40]. When taking the limit into the bulk of the GUE minors one obtains the bead process, see [18]. Further works on interlacing structures are [19, 32, 39, 56].

- *GUE minors and TASEP*: Both the GUE minor process and the antisymmetric one occurs in the diffusion scaling limit of TASEP [11, 12].

- 2+1 *dynamics*: The Markov process on interlacing structure introduced in [5] is not restricted to continuous time TASEP, but it is much more general. For example, it holds for PushASEP dynamics [14] and can be used for growth with a wall too [15]. In a discrete setting, a similar approach leads to a shuffling algorithm for boxed plane partitions [13]. As already mentioned, the connection between shuffling algorithm and interlacing particle dynamics is proven in [41] (the connection with discrete time TASEP is however not mentioned).

- $2+1$ *anisotropic growth*: In the large time limit in the $2+1$ growth model the Gaussian Free Field arises, see [5] or for a more physical description of the result [7]. In particular, height fluctuations live on a $\sqrt{\ln t}$ scale (in the bulk) and our model belongs to the anisotropic KPZ class, like the model studied in [45].

- *Interlacing and asymptotics of TASEP*: Large time asymptotics of TASEP particles' positions with a few but important types of initial condition have been worked out using the framework initiated with [9]. Periodic initial conditions are studied in [9] and for discrete time TASEP (sequential update [8], parallel update [10]). The limit process of the rescaled particles' positions is the Airy_1 process. For step

initial condition it was the Airy_2 process [29]. The transition process between these two has been discovered in [11], see also the review [23]. Finally, the above technique can be used also for non-uniform jump rates where shock can occur [12].

- *Line ensembles method and corner growth models*: TASEP can be also interpreted as a growth model, if the occupation variables are taken to be the discrete gradient of an interface. TASEP belongs to the so-called Kardar-Parisi-Zhang (KPZ) universality class of growth models. It is in this context that the first connections between random matrices and stochastic growth models have been obtained [27]. The model studied is analogue to step initial conditions for TASEP. This initial condition can be studied using non-intersection line ensembles methods [28, 29]. The Airy_2 process was discovered in [47] using this method, see also [31, 52, 55] for reviews on this technique. The non-intersecting line description is used also to prove the occurrence of the Airy_2 process at the edge of the frozen region in the Aztec diamond [30].

- *Stationary TASEP and directed percolation*: Directed percolation for exponential/geometric random variables is tightly related with TASEP. In particular, the two-point function of stationary TASEP can be related with a directed percolation model [46]. The large time behavior of the two-point function conjectured in [46] based on universality was proven in [24]. Some other universality-based of [46] conjectures have been verified in [1]. The large time limit process of particles' positions in stationary TASEP, the corresponding point-to-point directed percolation (with sources), and also for a related queuing system, has been unraveled in [3]. The different models shares the same asymptotics due to the slow-decorrelation phenomena [22].

- *Directed percolation and random matrices*: Directed percolation, the Schur process and random matrices have also nice connections; from sample covariance matrices [2], to small rank perturbation of Hermitian random matrices [50], and to the generalization [16].

B Christoffel-Darboux formula

Here we prove Christoffel-Darboux formula (6). First of all, we prove the three term relation (7). From $q_n(x)/u_n = x^n + \cdots$ it follows that

$$\frac{q_n(x)}{u_n} - \frac{xq_{n-1}(x)}{u_{n-1}} \tag{100}$$

is a polynomials of degree $n-1$. Thus,

$$\frac{q_n(x)}{u_n} = \frac{xq_{n-1}(x)}{u_{n-1}} + \sum_{k=0}^{n-1} \alpha_k q_k(x), \quad \alpha_k = \left\langle \frac{q_n}{u_n} - \frac{Xq_{n-1}}{u_{n-1}}, q_k \right\rangle_\omega, \tag{101}$$

where X is the multiplication operator by x, and $\langle f, g\rangle_\omega = \int_{\mathbb{R}} \omega(x) f(x) g(x) \mathrm{d}x$ is the scalar product.

Let us show that $\alpha_k = 0$ for $k = 0, \ldots, n-3$. Using $\langle Xf, g\rangle_\omega = \langle f, Xg\rangle_\omega$ we get

$$\alpha_k = \frac{1}{u_n}\langle q_n, q_k\rangle_\omega - \frac{1}{u_{n-1}}\langle q_{n-1}, Xq_k\rangle_\omega = 0 \tag{102}$$

for $k+1 < n-1$, since Xq_k is a polynomial of degree $k+1$ and can be written as linear combination of $q_0, \ldots, q_{k+1}$.

Consider next $k = n-2$. We have

$$\alpha_{n-2} = -\frac{1}{u_{n-1}}\langle q_{n-1}, Xq_{n-2}\rangle_\omega = -\frac{u_{n-2}}{u_{n-1}^2}, \tag{103}$$

because we can write

$$\begin{aligned} xq_{n-2}(x) &= u_{n-2}x^{n-1} + \text{a polynomial of degree } n-2 \\ &= \frac{u_{n-2}}{u_{n-1}} q_{n-1}(x) + \text{a polynomial of degree } n-2. \end{aligned} \tag{104}$$

Therefore, setting $B_n = \alpha_{n-1}u_n$, $A_n = u_n/u_{n-1}$, and $C_n = u_n u_{n-2}/u_{n-1}^2$, we obtain the three term relation (7). We rewrite it here for convenience,

$$q_n(x) = (A_n x + B_n) q_{n-1}(x) - C_n q_{n-2}(x). \tag{105}$$

From (105) it follows

$$\begin{aligned} &q_{n+1}(x)q_n(y) - q_n(x)q_{n+1}(y) \\ &\quad = A_{n+1} q_n(x) q_n(y)(x-y) + C_{n+1}\left(q_n(x)q_{n-1}(y) - q_{n-1}(x)q_n(y)\right). \end{aligned} \tag{106}$$

We now consider the case $x \neq y$. The case $x = y$ is obtained by taking the $y \to x$ limit. Dividing (106) by $(x-y)A_{n+1}$ we get, for $k \geq 1$,

$$q_k(x)q_k(y) = S_{k+1}(x,y) - S_k(x,y), \tag{107}$$

where we defined

$$S_k(x,y) = \frac{u_{k-1}}{u_k} \frac{q_k(x)q_{k-1}(y) - q_{k-1}(x)q_k(y)}{x-y}. \tag{108}$$

Therefore (for $x \neq y$)

$$\sum_{k=0}^{N-1} q_k(x)q_k(y) = S_N(x,y) - S_1(x,y) + q_0(x)q_0(y) = S_N(x,y). \tag{109}$$

The last step uses $q_0(x) = u_0$ and $q_1(x) = u_1 x + c$ (for some constant c), from which it follows $q_0(x)q_0(y) = S_1(x,y)$. This ends the derivation of the Christoffel-Darboux formula.

C Proof of Proposition 6

Here we present the details of the proof of Proposition 6 since it shows how the choice of the orthogonal polynomial is convenient. The basic ingredients of the proof of Theorem 8 are the same, with the only important difference that the functions in the determinants in (17) are not yet biorthogonal.

First of all, let us verify the two relations (15). We have

$$\int_{\mathbb{R}} K_N^{\mathrm{GUE}}(x,x)\mathrm{d}x = \sum_{k=0}^{N-1} \langle q_k, q_k \rangle_\omega = N, \tag{110}$$

and

$$\begin{aligned} \int_{\mathbb{R}} K_N^{\mathrm{GUE}}(x,z)K_N^{\mathrm{GUE}}(z,y)\mathrm{d}z &= \sum_{k,l=0}^{N-1} \sqrt{\omega(x)\omega(y)} q_k(x)q_l(y)\langle q_k, q_l \rangle_\omega \\ &= K_N^{\mathrm{GUE}}(x,y). \end{aligned} \tag{111}$$

By Lemma 5, Equation (12), and the definition of K_N^{GUE}, we have

$$\begin{aligned} &\rho_{\mathrm{GUE}}^{(n)}(x_1,\dots,x_n) \\ &\quad = c_N \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} \det\left[K_N^{\mathrm{GUE}}(x_i,x_j)\right]_{1\leq i,j\leq N} \mathrm{d}x_{n+1}\dots\mathrm{d}x_N. \end{aligned} \tag{112}$$

We need to integrate $N-n$ times, each step is similar. Assume therefore that we already reduced the size of the determinant to $m \times m$, i.e., integrated out $x_{m+1}, \ldots, x_N$. Then, we need to compute

$$\int_{\mathbb{R}} \det \left[K_N^{\mathrm{GUE}}(x_i, x_j) \right]_{1 \le i,j \le m} \mathrm{d}x_m. \tag{113}$$

In what follows we write only K instead of K_N^{GUE}. We develop the determinant along the last column and get

$$\begin{aligned} \det \left[K(x_i, x_j) \right]_{1 \le i,j \le m} &= K(x_m, x_m) \det \left[K(x_i, x_j) \right]_{1 \le i,j \le m-1} \\ &\quad + \sum_{k=1}^{m-1} (-1)^{m-k} K(x_k, x_m) \det \begin{bmatrix} \left[K(x_i, x_j) \right]_{1 \le i,j \le m-1, i \ne k} \\ \left[K(x_m, x_j) \right]_{1 \le j \le m-1} \end{bmatrix} \\ &= K(x_m, x_m) \det \left[K(x_i, x_j) \right]_{1 \le i,j \le m-1} \\ &\quad + \sum_{k=1}^{m-1} (-1)^{m-k} \det \begin{bmatrix} \left[K(x_i, x_j) \right]_{1 \le i,j \le m-1, i \ne k} \\ \left[K(x_k, x_m) K(x_m, x_j) \right]_{1 \le j \le m-1} \end{bmatrix}. \end{aligned} \tag{114}$$

Finally, by using the two relations (15), Equation (113) becomes

$$\begin{aligned} &N \det \left[K(x_i, x_j) \right]_{1 \le i,j \le m-1} + \sum_{k=1}^{m-1} (-1)^{m-k} \det \begin{bmatrix} \left[K(x_i, x_j) \right]_{1 \le i,j \le m-1, i \ne k} \\ \left[K(x_k, x_j) \right]_{1 \le j \le m-1} \end{bmatrix} \\ &= (N - (m-1)) \det \left[K(x_i, x_j) \right]_{1 \le i,j \le m-1}. \end{aligned} \tag{115}$$

This result, applied for $m = N, N-1, \ldots, n+1$, leads to

$$\rho_{\mathrm{GUE}}^{(n)}(x_1, \ldots, x_n) = c_N N! \det \left[K_N^{\mathrm{GUE}}(x_i, x_j) \right]_{1 \le i,j \le n}. \tag{116}$$

Now we need to determine c_N. Since c_N depends only of N, we can compute it for the $n = 1$ case. From the above computations, we have $\rho_{\mathrm{GUE}}^{(1)}(x) = c_N N! K_N^{\mathrm{GUE}}(x, x)$ and $\int_{\mathbb{R}} \rho_{\mathrm{GUE}}^{(1)}(x) \mathrm{d}x = N$ we have $c_N = 1/N!$.

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